Kaluza-Klein Theory

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Abstract

This paper deals with Kaluza-Klein theory, and its incarnations. The general framework is outlined, as well as an introduction to the compactified, projective, and noncompactified approaches. While there is some theoretical background, an emphasis is made on the qualititative features of each style. Finally, Kaluza-Klein theory is discussed in terms of its relation to higher-dimensional theories, including superstring theory.

1 Introduction

Unifying forces and adding extra dimensions have proved irresistable to physicists over the years. Electricity and magnetism were unified by Maxwell. Minkowski showed that time can be interpreted as a fourth dimension in Einstein's special theory of relativity. The calculational convenience and mathematical elegance is a truly compelling reason for adding higher dimensions to theories. Often phenomena difficult to explain in three dimensions can be shown to be part of a simpler theory in higher dimensions.

By the early 1920's, electromagnetism and general relativity were both experiencing great success. Naturally, the next step would be to try to unify these two forces. In 1921, Theodor Kaluza tried this. By adding a fifth dimension to spacetime, he was able to demonstrate that his theory contained four-dimensional general relativity in the prescence of an electromagnetic field. In doing so, he unified not just gravity and electromagnetism, but also geometry and matter. This was a remarkable result, but where was the extra dimension? Why had it not been observed? In 1926, Oscar Klein proposed that the fifth dimension is curled up in a circle of very small radius. Hence, the concept of compactification was born, and it was proposed that extra dimensions would not be observable except on very high energy scales. These energy scales are currently inaccessable.

Modern Kaluza-Klein theories have three main features:

1. They treat nature as pure geometry. That is, the grvatitational and electromagnetic fields are completely contained in a five-dimensional Einstein tensor, i.e. in a metric and its derivatives. Kaluza-Klein theory thus achieves not only a unification of gravity and electromagnetism, but also of matter and geometry.

2. Kaluza-Klein theories are minimal extensions of general relativity. This means there is no modification to Einstein's theory other than the equations run from 0 to 4 instead of 0 to 3.

3. The theory is cylindrical, which means that the physics depends only on the first four coordinates.

The first two seem to be natural and desirable from an Occam's razor point of view, but the third seems to be somewhat ad hoc. Higher dimensional theories have tried to do away with this third feature in a variety of ways. However, each one sacrifices another one of these main feautures as well. There are three main approaches for doing this which will be discussed here - compactified, projective, and noncompactified. However, first some of the mathematical structure of the theory will be introduced.

2 Mathematical Structure

The Einstein equations in a five dimensional vacuum are:

$$\hat{G}_{AB} = 0 \tag{1}$$

which can also be represented as:

$$\hat{R}_{AB} = 0 \tag{2}$$

where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \hat{R}\hat{g}_{AB}/2$ is the five-dimensional Einstein tensor, and $\hat{R}_{AB}$ and $\hat{R} = \hat{g}_{AB}R^{AB}$ five-dimensional Ricci tensor and Ricci scalar. The five-dimensional metric tensor is denoted as $\hat{g}_{AB}$. Note that any mathematical entity with a "hat" symbol denotes a five-dimensional quantity. Not surprisingly, these equations can also be obtained by setting the variation with respect to the five-dimensional metric of the following five-dimensional Einstein-Hilbert action to zero:

$$S = -\frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}d^4xdy \tag{3}$$

where $y \equiv x^4$ represents the extra fifth dimension. Note that there are no matter sources in these equations which embodies the feature (i) of modern Kaluza-Klein theories. Another desirable feature for Kaluza-Klein theory is to have it as a minimal extension of general relativity, feature (ii). First we define the five-dimensional Ricci tensor and Christoffel symbols (which are exactly the same in four-dimensions except for the indicies run from 0 to 3 as opposed to 0 to 4).

$$\hat{R}_{AB} = \partial_C\hat{\Gamma}^C_{AB} - \partial_B\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC} \tag{4}$$

$$\hat{\Gamma}^C_{AB} = \frac{1}{2}\hat{g}^{CD}\left(\partial_A\hat{g}_{DB} + \partial_B\hat{g}_{DA} - \partial_D\hat{g}_{AB}\right) \tag{5}$$

Now, we just need to define the five-dimensional metric. It is more or less defined as the four-dimensional metric tensor, $g_{\alpha\beta}$ with the electromagnetic potential, A_α, and a scalar field, ϕ, the dilaton. It is then defined as,

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2 A_\alpha A_\beta & \kappa^2\phi^2 A_\alpha \\ \kappa^2\phi^2 A_\beta & \phi^2 \end{pmatrix} \tag{6}$$

where the particular form is simply for convenience. The constant, κ, will give us the right multiplicative factors later on. Here, the Greek indicies α,β run from 0 to 3. The four-dimensional metric signature is taken to be $(+\ -\ -\ -)$, and we set $c = 1$, $\hbar = 1$, and $G = 1$. The general mathematical structure is essentially complete, and we are now able to discuss some of the more specific aspetcs of Kaluza-Klein theory.

2.1 The Cylinder Condition

The third feature (iii) of Kaluza Klein theory, the cylinder condition, says to drop all derivatives with respect to the fifth coordinate. By doing this, one finds from the five-dimensional Einstein equations (requiring use of eq.(2),(4).(5), and (6)), the following field equations in four dimensions:

$$G_{\alpha\beta} = \frac{\kappa^2\phi^2}{2}T^{EM}_{\alpha\beta} - \frac{1}{\phi}\left[\nabla_\alpha(\partial_\beta\phi) - g_{\alpha\beta}\Box\phi\right] \tag{7}$$

$$\nabla^\alpha F_{\alpha\beta} = -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta} \tag{8}$$

$$\Box\phi = \frac{\kappa^2\phi^3}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{9}$$

where

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - Rg_{\alpha\beta}/2 \tag{10}$$

is the four-dimensional Einstein tensor,

$$T^{EM}_{\alpha\beta} \equiv g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}/4 - F^{\gamma}_{\alpha}F_{\beta\gamma} \tag{11}$$

is the electromagnetic stress-energy tensor, and

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{12}$$

is the electromagnetic field tensor. These represent a total of 15 equations, as expected since there are 15 independent elements in the five-dimensional metric. Furthermore, if one says that the scalar field ϕ is constant, then eq.(7) and (8) reduce to:

$$G_{\alpha\beta} = 8\pi G\phi^2 T^{EM}_{\alpha\beta} \tag{13}$$

$$\nabla^\alpha F_{\alpha\beta} = 0 \tag{14}$$

which are just the standard four-dimensional Einstein and Maxwell equations, hence gravity and electromagnetism are explicitily contained. Note that the parameter, κ, has been identified in terms of the gravitational constant, G, by

$$\kappa = 4\sqrt{\pi G}. \tag{15}$$

By setting ϕ equal to a constant, one must note that eq.(9) neccesarily implies that $F_{\alpha\beta}F^{\alpha\beta} = 0$. Today, it is common to do an equivalent derivation using the following action

$$S = -\int d^4x\sqrt{-g}\phi\left(\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\alpha\beta}F^{\alpha\beta} + \frac{2}{3\kappa^2}\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi^2}\right). \tag{16}$$

Thus, we have demonstrated how the sourceless field equations (2) lead to fields with source matter. This demonstrates how four-dimensional matter (at least in the form of electromagnetic radiation) can arise purely from a higher dimensional vacuum. Although we have presented how four-dimensional gravity and electromagnetism come from Kaluza Klein theory, there are two totally unjustified aspects so far (assuming that we totally believe general relativity)- the cylinder condition (setting the derivatives with respect to the fifth dimension to be zero) and setting ϕ equal to a constant. We have not really accomplished that much from my point of view other than a clever mathematical trick with some unjustified assumptions. These assumptions will be explored further in the rest of the paper.

3 Brans-Dicke Theory

Setting ϕ to a constant gives some nice mathematical results that correspond to our common view of our four-dimensional world. But why do this? If one does not set ϕ equal to a constant, one can arrive at a special case of Brans-Dicke theory which is a competing theory with general relativity which says that the gravitational interaction is mediated by a scalar field as well as a tensor field. We can more easily see this by setting $A_\alpha = 0$. Without the cylinder condition, this does not give any loss of generality, but with it, this is in effect choosing a special set of coordinates which is no longer invariant to general five-dimensional coordinate systems. Nonetheless, we continue on using the three main features of Kaluza-Klein theory (including the cylinder condition). Our five-dimensional metric (6) reduces to:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \phi^2 \end{pmatrix} \tag{17}$$

Using this, the action reduces to a special case of the Brans-Dicke action:

$$S_{BD} = -\int d^4x\sqrt{-g}\left(\frac{R\phi}{16\pi G} + \omega\frac{\partial^\alpha\phi\partial_\alpha\phi}{\phi}\right) + S_m \tag{18}$$

where $\omega = 0$ and there is no S_m or actions associated with a matter field. However, currently the value of ω is constrained to be on the order of 500, so the model presented here is not really viable in the current epoch. There are some ways to get around this, but it is nonetheless interesting how Kaluza-Klein theory contains aspects of Brans-Dicke theory. Brans-Dicke theory apparently is in agreement with Einstein's general relativity, but the observations have considerably constrained the allowed parameters, thus making general relativity the more popular and useful theory.

4 Compactification

We now discuss Klein's compactification mechanism, which is not exactly the same as the clyinder condition. In fact, everything discussed so far has essentially been Kaluza's theory and strictly speaking, Kaluza-Klein theory is Kaluza's theory with the following compactification mechanism. This compactification mechanism gets rid of the cylinder condition (a mathematical trick) and simply says that the observed lack of dependence on the fifth dimension is because it is exceedingly small (a physical trick). He assumed that it was lengthlike with a circular topology and a small scale. The circular topology implies that it is periodic, such that $f(x,y) = f(x, y + 2\pi r)$ where $x = (x^0, x^1, x^2, x^3)$, $y = x^4$, and r is the scale or radius of the fifth dimension. Because of this, we can Fourier expand all of the fields:

$$g_{\alpha\beta}(x,y) = \sum_{n=-\infty}^{n=\infty} g_{\alpha\beta}^{(n)}(x)e^{iny/r} \tag{19}$$

$$A_\alpha(x,y) = \sum_{n=-\infty}^{n=\infty} A_\alpha^{(n)}(x) e^{iny/r} \tag{20}$$

$$\phi(x,y) = \sum_{n=-\infty}^{n=\infty} \phi^{(n)} e^{iny/r} \tag{21}$$

where the superscript $^{(n)}$ refers to the n^{th} Fourier mode. These modes carry a momentum in the y-direction on the order of $|n|/r$. Thus, if r is small enough, the y-momenta of any non-zero mode will be so large that they will be experimentally inaccessable. Only the $n = 0$ mode which is independent of y will be observable. This unobserved dimension is currently constrained to be smaller than 10^{-18} m, and theorists often set r equal to the Planck length which is about 10^{-35} m.

5 Projective Theories

An alternative to Klein's compactification mechanism was introduced by Veblen and Hoffman in 1931. They showed that the fifth dimension could be "absorbed into" four-dimnensional spacetime if one replaced the affine tensors of general relativity with projective ones. This essentially results in the fifth coordinate being regarded as a visual aid as opposed to really being a new coordinate. Since the new dimension was not really real, then there would be need to explain why physics clearly did not depend on the fifth coordinate. However, this approach required that one alter the geometrical foundation of the theory. This paper will not go into detail of the projective theories, but they are not seen to be an improvement over the compactification mechanism.

6 Noncompactified Theories

An alternative way of understanding the fifth dimension, and perhaps the most natural, is to simply take it at face value, and assume that nature is only approximately independent of them. This is analogous to nature being Newtonain except at highly relativistic speeds. One must ask why nature should still appear to be so cylindrical (i.e. independent of the fifth coordinate). One idea to explain this is to say that particles are trapped near a four-dimensional hypersurface by a potential well, although this is really no improvement over compactification. So far, we have assumed that the fifth dimension is lengthlike, but we can say this is not necessarily the case. In this case, the explanation for the independence of nature on the fifth coordinate is to be found in the physical interpretation of the extra coordinates. For example, the first proposal of this kind was the "space-time-mass" theory by Wesson in 1983. He suggested that a fifth dimension might be associated with rest mass by the equation

$$x^4 = Gm/c^2. \tag{22}$$

In this theory, the rest mass would vary with time. However, this variation is quite small and consistent with experiment. Nonetheless, the importance of the noncompactified theories is that physics is allowed to depend on the extra dimension, quite unlike the compactified theories or projective theories. To arrive at the field equations, one follows the exact same procedure as Kaluza except now we keep any derivitives with respect to the fifth dimension. We again use eq.(4) and (5) and the following metric:

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \epsilon\phi^2 \end{pmatrix} \tag{23}$$

where we include ϵ to allow for a timelike or spacelike signature for the fifth dimension (i.e. $\epsilon = 1$ or -1). Note also that since we are not imposing the clyinder condition we have chosen coordinates such that $A_\alpha = 0$, which entails no loss of generality. We also impose that $\hat{R}_{AB} = 0$, which says that there is no higher-dimensional matter. This recipe gives a rather complicated form for the Ricci tensor in five-dimensions (as well as four dimensions). One requires that Einstein's equations in four-dimensions with matter hold in this noncompactified approach:

$$8\pi T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} \tag{24}$$

The details are not presented here but it can be shown that the four-dimensional Einstein equations, $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ are automatically contained in the five-dimensional vacuum equations

$$\hat{G}_{AB} = 0 \tag{25}$$

in the form of

$$8\pi G T_{\alpha\beta} = \frac{\nabla_\beta(\partial_\alpha\phi)}{\phi} - \frac{\epsilon}{2\phi^2}\left[\frac{\partial_4\phi\partial_4 g_{\alpha\beta}}{\phi} - \partial_4(\partial_4 g_{\alpha\beta}) + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta}\right.$$
$$\left. - \frac{g^{\gamma\delta}\partial_4 g_{\gamma\delta}\partial_4 g_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\left(\partial_4 g^{\gamma\delta}\partial_4 g_{\gamma\delta} + (g^{\gamma\delta}\partial_4 g_{\gamma\delta})^2\right)\right]. \tag{26}$$

The matter described in this manner comes from pure geometry in the five-dimensional world. This is called the "induced-matter interpretation" of Kaluza-Klein theory. This matter contained in this four-dimensional $T_{\alpha\beta}$ is apparently general enough to describe many physical systems.

7 Extension to $D > 5$ Theories

A general overview of five-dimensional Kaluza-Klein theories have been presented so far. However, only gravity and electromagnetism have been explicitily included in the formalism. What about the other two forces? Well, the natural idea was to add more dimensions. Edward Witten proved in 1981 that eleven was the minimum number of dimensions required by Kalua-Klein theory to unify all the forces of the standard model of particle physics. Also, in 1980, Freund and Rubin showed that compactification of an 11-dimensional model could occur with either four or seven compact dimensions. Reality obviously leads one to believe there are seven compact dimensions. This theory called 11-dimensional supergravity came to be a leading candidate for the fabled "theory of everything."

However, there were difficulties associated with the 11-dimensional theory that could be fixed by making the theory 10-dimensional. However, there was a problem of uniqueness with the 10-dimensional theory and new anomalies were discovered. Finally, a breakthrough was made by Green and Schwarz with the Green-Schwarz mechanism which showed that only two 10-dimensional supergravity models could make all the anamolies vanish. They were those based on the groups $SO(32)$ and $E_8 \times E_8$. Extra terms had to be added to the Lagrangian for these two theories but they were not totally arbitrary as they were terms for a low-energy approximation to superstring theory.

At this time, he newest compactified Kaluza-Klein theories were shifting from supergravity to superstring theories. The Green-Schwarz breakthrough caused interest in superstring theories to explode. There was still the uniqueness problem though in that the two allowable groups allowed five different string theories among them. However, Witten showed that it was possible to view these five 10-dimensional string theories as aspects of an 11-dimensional so-called M-theory.

8 Conclusion

Kaluza-Klein theory was the first significant attempt to go to more dimensions than are observed. By letting the indices run from 0 to 4, he succeeded in unifying general relativity and electromagnetism into a single theory. Klein suggested that the fifth dimension should be compactified to such a small size that observable physics would almost certainly never see any effects of it. This, in the strict sense of the words, is Kaluza-Klein theory. Modifications were made that included a projective approach and noncompactified approach. It is, at this point, impossible to say which one really describes reality, as all three are consistent with experiment. The mathematical trick of throwing on the electromagnetic potential (and a scalar field) onto the metric is far away from a complete theory describing the universe. Kaluza-Klein theory is a precursor to truly fundamental higher-dimensional theories. It seems that, if a theory of everything is ever found, it will almost certainly be one that lies in more than four dimensions. Kaluza-Klein theory was the first glimpse into this type of universe.

9 References

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