

A Simple Stellar Model

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1 Modeling the Structure of a Star

We can effectively model the structure of a star with the following equations. We start out with the **mass conservation equation**:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho. \quad (1)$$

This equation states how the interior mass of a star must change with distance from the center. Here, M_r is the interior mass at a certain radius r , while ρ is the mass density. The next equation is based on the condition of **hydrostatic equilibrium**. This says that there must be a pressure gradient in the star to counteract the force of gravity.

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \quad (2)$$

The variable P is for pressure, while G is the gravitational constant we all know and love. The next equation is for the condition of **radiative equilibrium**, where T is the temperature.

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho}{T^3} \frac{L_r}{4\pi r^2} \quad (3)$$

Its sister equation is one for **convective equilibrium**.

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr} \quad (4)$$

The condition of radiative equilibrium will be used for part of the star while convective equilibrium will be used for a different part. This will be further discussed later. The last equation needed for this model is the equation of state,

$$P = \frac{k}{\mu H} \rho T, \quad (5)$$

where k is Boltzmann's constant, H is the mass of a proton, and μ is the mean molecular weight which is defined as

$$\frac{1}{\mu} = 2X + \frac{3}{4}Y + \frac{1}{2}Z, \quad (6)$$

where X, Y, and Z are the relative abundances of hydrogen, helium, and the heavy elements respectively. Finally, we have the boundary conditions at the center and the surface:

$$center : r = 0, \quad M_r = 0; \quad surface : r = R, \quad T = 0, \quad P = 0.$$

2 Preparing the Model for Numerical Analysis

First, we want to switch to non-dimensional variables. We make the following transformations:

$$M_r = qM \quad r = xR \quad P = p \frac{GM^2}{4\pi R^4} \quad T = t\mu \frac{H}{k} \frac{GM}{R}.$$

Using these transformations, Eqs.(1),(2),(3), and (4) are changed into the following forms:

$$\frac{dq}{dx} = \frac{px^2}{t} \tag{7}$$

$$\frac{dp}{dx} = -\frac{pq}{tx^2} \tag{8}$$

$$\frac{dt}{dx_{rad}} = -\frac{Cp^{1.75}}{x^2 t^{8.25}} \tag{9}$$

$$\frac{dt}{dx_{conv}} = -\frac{2}{5} \frac{q}{x^2}, \tag{10}$$

now with the boundary condition that at the surface ($x = 1$), $p = t = 0$ and $q = 1$, along with making sure that $q \rightarrow 0$ as $x \rightarrow 0$. Notice that the constant C in Eq.(9) has the value 3.04271×10^{-6} to satisfy all of these conditions. Also, note that at the interface of radiative and convective equilibrium, we have the condition that

$$\frac{dt}{dx_{rad}} = \frac{dt}{dx_{conv}}.$$

The next step is to change our current equations into ones with logarithmic variables. Because these variables have such a wide range of values, logarithmic variables effectively put them on a much narrower scale, so standard precision variables can be used in a program. We have the following transformations:

$$g_1 = \log p \quad g_2 = \log q \quad g_3 = \log t \quad y = \log x.$$

Using these transformation, it will be left as an exercise to the reader to show that Eqs.(7),(8),(9), and (10) become the following respectively:

$$\frac{dg_2}{dy} = 10^{g_1 - g_2 - g_3 + 3y} \tag{11}$$

$$\frac{dg_1}{dy} = -10^{g_2 - g_3 - y} \tag{12}$$

$$\frac{dg_3}{dy} = -C10^{1.75g_1 - 9.25g_3 - y} \quad (13)$$

$$\frac{dg_3}{dy} = -\frac{2}{5}10^{g_2 - g_3 - y} \quad (14)$$

These are the equations that will be used in a program implementing the 4th order Runge-Kutta method.

3 Runge-Kutta Method

The Runge-Kutta method for integrating N differential equations is a simple, yet effective approach. Using an initial condition, the algorithm moves a small step along the equations to find the next point. The algorithm to do this can be summarized as follows:

$$\begin{aligned} k_1 &= hf'(x_n, y_n) \\ k_2 &= hf'\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf'\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf'(x_n + h, y_n + k_3), \end{aligned}$$

where h is the step size. Now these values are used to calculate the next point which is given by the following equation:

$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5). \quad (15)$$

We obtain our initial conditions for our logarithmic variables by using $x = 1, q = 1, t = 0$, and $p = 0$. However, since the logarithm of 0 is negative infinity, we use the following formulas for the initial conditions of t and p , using an x very close to 1 (I used 0.99).

$$p^{1.75} = \frac{1.75}{8.25} \frac{1}{C} t^{8.25}$$

$$t = \frac{1.75}{8.25} \left(\frac{1}{x} - 1\right).$$

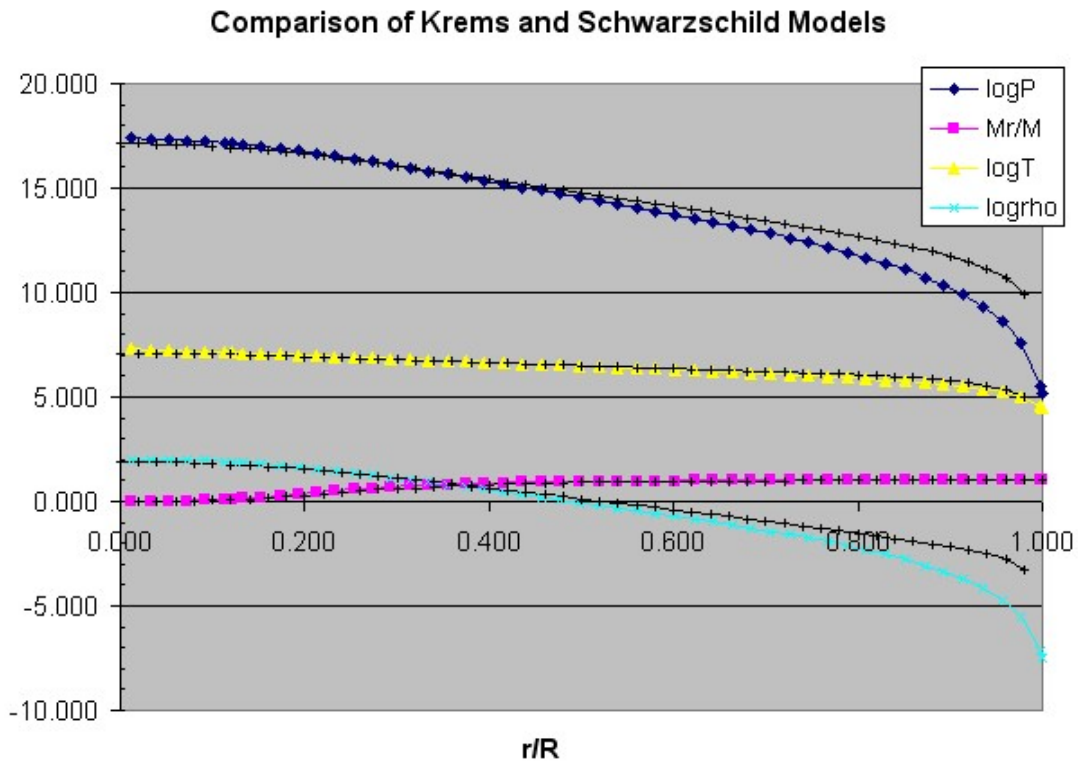
Plugging in the value of 0.99 for x and 3.04271×10^{-6} for C , we find that $t = 0.002143$ and $p = 1.53089 \times 10^{-10}$. Converting these numbers into our logarithmic variables gives us the final initial conditions to be implemented into the program:

$$g_1 = -9.81506 \quad g_2 = 0 \quad g_3 = -2.669054 \quad y = 0.$$

4 The Program and Results

Runge-Kutta method was employed in Fortran 77 (shown in Appendix B) to integrate the three equations. In this program, we start at the surface and integrate in so the step size is negative. Notice that we switch from radiative to convective equilibrium at the interface when the two are equal (which is effectively at the point where their ratio switches from a number less than one to a number greater than one). Here, we model our sun, assuming the chemical composition of the sun is $X = 0.68, Y = 0.30,$ and $Z = 0.02$.

After the successful running of the program, values very close to those found by M. Schwarzschild are obtained (see Appendix A). In the program, we switch from radiative to convective energy transport at $x \approx 0.122$. This is at a temperature of about 10^7 K and a mass density of about 80 g/cm^3 . In addition to the data in Appendix A, one can how close my values are to Schwarzschild's by viewing the following graph. Schwarzschild's data is seen as the black lines, while my data is shown as the color lines.



In conclusion, we have accomplished a successful model of our sun to a first approximation. This simple model is certainly not the whole story. There are much more complicated models that have been developed that describe the more complex features of a stellar structure. Nonetheless, this simple model is an effective tool for learning about stellar structures.