Electricity and Magnetism

Special Relativity

$s_{12}^2 = s_{12}'^2 = c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2$
$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
timelike: $s_{12}^2 > 0$ events cannot be simulataneoues in any frame
spacelike: $s_{12}^2 < 0$ events cannot occur at same point in any frame
proper time: $\tau = \frac{ds}{c} = \gamma dt$ $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ $\beta = \frac{v}{c}$
$x = \gamma(x' + vt')$ $y = y'$ $z = z'$ $t = \gamma\left(t' + \frac{vx'}{c^2}\right)$
$v_x = \frac{v_x'+V}{1+v_x'v/c^2}$ $v_y = \frac{v_y'\sqrt{1-v^2/c^2}}{1+v_x'v/c^2}$ $v_z = \frac{v_z'\sqrt{1-v^2/c^2}}{1+v_x'v/c^2}$
general $\vec{v}$: $u_\parallel = \frac{u_\parallel'+v}{1+\vec{v}\cdot\vec{u}'/c^2}$ $\vec{u}_\perp = \frac{\vec{u}_\perp'}{\gamma_v(1+\vec{v}\cdot\vec{u}'/c^2)}$
length contraction: $l = l'/\gamma$ $vol = vol'/\gamma$ $d^3xdt = d^3x'dt'$
time dilation: $t = \gamma t'$
$\vec{a}_\parallel = \frac{(1-\frac{v^2}{c^2})^{3/2}}{(1+\frac{\vec{v}\cdot\vec{u}'}{c^2})^3}\vec{a}_\parallel'$ $\vec{a}_\perp = \frac{1-\frac{v^2}{c^2}}{(1+\frac{\vec{v}\cdot\vec{u}'}{c^2})^3}\left(\vec{a}_\perp' + \frac{\vec{v}}{c^2}\times(\vec{a}'\times\vec{u}')\right)$
doppler: $k_0' = \gamma(k_0 - \vec{\beta}\cdot\vec{k})$ $k_\parallel' = \gamma(k_\parallel - \beta k_0)$ $\vec{k}_\perp' = \vec{k}_\perp$
light: $\omega' = \gamma\omega(1-\beta\cos\theta)$ $\tan\theta = \frac{\sin\theta}{\gamma(\cos\theta-\beta)}$
$\phi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$

4-vectors

$x^i = (ct, \vec{x})$ $x_i = (ct, -\vec{x})$ $k^i = (2\pi/\lambda, \vec{k})$ $u^i = (\gamma c, \gamma\vec{v})$
scalar: $\bar{\psi}(x^i) = \psi(x^i)$ covariant 4-vector: $\bar{B}_j = B_k\frac{\partial x^k}{\partial x^j}$
2^{nd} rank tensor: $\bar{A}_{ij} = A_{kl}\frac{\partial x^k}{\partial x^i}\frac{\partial x^l}{\partial x^j}$
metric tensor: $g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
Lorentz transformation: $\Gamma^\mu_{\nu'} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Relativistic Mechanics

$u^i = c\frac{dx^i}{ds} = (\gamma c, \gamma\vec{v})$ $u_iu^i = c^2$ $\vec{F} = \frac{d\vec{p}}{dt}$
$p^i = mu^i = (\gamma mc, \gamma m\vec{v}) = (\frac{E}{c}, \vec{p})$ $p^ip_i = \frac{E^2}{c^2} - p^2 = m^2c^2$
$L = -mc^2\sqrt{1-v^2/c^2}$ $E = H = \vec{p}\cdot\vec{v} - L = \frac{mc^2}{\sqrt{1-v^2/c^2}}$
$\frac{dE}{dt} = \vec{f}\cdot\vec{v}$ $\vec{p} = \frac{E}{c^2}\vec{v}$ $E^2 = p^2c^2 + m^2c^4$
total 4-momentum conserved (energy and momentum)
$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{p_i}$ $\frac{\partial f}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial f}{\partial t}, \vec{\nabla}f\right)$
$A_0' = \gamma(A_0 - \vec{\beta}\cdot\vec{A})$ $A_\parallel' = \gamma(A_\parallel - \beta A_0)$ $\vec{A}_\perp' = \vec{A}_\perp$
Compton: $\lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$ $v = \nu\lambda$ $k = \frac{2\pi}{\lambda}$ $\omega = 2\pi\nu$

Electrodynamics

$L = -mc^2\sqrt{1-v^2/c^2} - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$
$A^i = (\phi, \vec{A})$ $j^i = \sum_\alpha e_\alpha\delta(\vec{r}-\vec{r}_\alpha)\frac{dx^i_\alpha}{dt} = (\rho c, \vec{J})$
$\vec{p} = \frac{\partial L}{\partial\vec{v}} = \gamma m\vec{v} + \frac{q}{c}\vec{A}$
$E = \underbrace{\gamma mc^2}_{E'} + q\phi = H = \sqrt{(\vec{p}-\frac{q}{c}\vec{A})^2c^2 + m^2c^4} + q\phi$
$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \vec{\nabla}\times\vec{A}$ $\vec{F} = q\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right]$
$\frac{dE'}{dt} = \vec{f}\cdot\vec{v} = e\vec{v}\cdot\vec{E}$ $\frac{\partial L}{\partial t} = 0 \rightarrow$ energy conserved

Gauage Invariance: $A_i' = A_i - \frac{\partial A_i}{\partial x^i}$
Adiabatic Invariant for $\vec{B} = \hat{z}B(t)$: $\mu = \frac{mv_\perp^2}{2B}$ $\Omega_c = \frac{eB_0}{\gamma mc}$
Weakly Inhomogeneous $\vec{B}$: $\vec{v}_D = \frac{c}{B}\left[-\frac{\mu}{e}\vec{\nabla}_\perp|B| + \frac{mv_\parallel}{e}\frac{\hat{R}}{R}\right]\times\hat{B}$
$\frac{d}{dt}(mv_\parallel) = -\mu\hat{B}\cdot\vec{\nabla}B$

Electromagnetic Field Tensor

$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$
$\frac{dp^i}{d\tau} = mc\frac{du^i}{ds} = q\frac{F^{ik}u_k}{c}$
$F_{ij}F^{ij} = 2(B^2 - E^2)$ $e^{iklm}F_{ik}F_{lm} = -4\vec{E}\cdot\vec{B}$
Maxwell: $\partial_aF^{ab} = \frac{4\pi}{c}J^b$ $\partial_aF_{bc} + \partial_bF_{ca} + \partial_cF_{ab} = 0$

Transformation of Fields

$E_x' = E_x$ $\vec{E}_\perp' = \gamma[\vec{E}_\perp + \vec{\beta}\times\vec{B}_\perp]$
$B_x' = B_x$ $\vec{B}_\perp' = \gamma[\vec{B}_\perp - \vec{\beta}\times\vec{E}_\perp]$
if $|E| < |B|$ and $\vec{E}\cdot\vec{B} = 0$, frame with $\vec{v} = \frac{\vec{E}\times\vec{B}}{B^2}$ has no E
if $|E| > |B|$ and $\vec{E}\cdot\vec{B} = 0$, frame with $\vec{v} = \frac{\vec{E}\times\vec{B}}{E^2}$ has no B

Maxwell's Equations

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$	$\oint_S\vec{E}\cdot d\vec{A} = 4\pi\int_V\rho dV$
$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$	$\oint\vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_A\frac{d}{dt}(\vec{B}\cdot d\vec{A})$
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint\vec{B}\cdot d\vec{A} = 0$
$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$	$\oint\vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int_A\vec{J}\cdot d\vec{A} + \frac{1}{c}\int_A\frac{d}{dt}(\vec{E}\cdot d\vec{A})$
$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$	$\oint_S\vec{D}\cdot d\vec{A} = 4\pi\int_V\rho_f dV$
$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$	$\oint\vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_A\frac{d}{dt}(\vec{B}\cdot d\vec{A})$
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint\vec{B}\cdot d\vec{A} = 0$
$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J}_f + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$	$\oint\vec{H}\cdot d\vec{l} = \frac{4\pi}{c}\int_A\vec{J}_f\cdot d\vec{A} + \frac{1}{c}\int_A\frac{d}{dt}(\vec{D}\cdot d\vec{A})$

$\vec{D} = \vec{E} + 4\pi\vec{P}$	$\rho_b = -\vec{\nabla}\cdot\vec{P}$	$D_1^\perp - D_2^\perp = \sigma_f$
$\vec{H} = \vec{B} - 4\pi\vec{M}$	$\sigma_b = \hat{n}\cdot\vec{P}$	$E_1^\parallel = E_2^\parallel$
$\vec{P} = \chi_e\vec{E}$ $\vec{D} = \epsilon\vec{E}$	$\vec{J}_b = c\vec{\nabla}\times\vec{M}$	$B_1^\perp = B_2^\perp$
$\vec{M} = \chi_m\vec{H}$ $\vec{H} = \frac{1}{\mu}\vec{B}$	$\vec{K}_b = -c\hat{n}\times\vec{M}$	$\vec{H}_1^\parallel - \vec{H}_2^\parallel = \vec{K}_f\times\hat{n}$

$\vec{F} = q(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B})$ $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ $\vec{B} = \vec{\nabla}\times\vec{A}$
continuity: $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 = \frac{\partial j^i}{\partial x^i}$ $\frac{dQ}{dt} + \oint\vec{J}\cdot d\vec{A} = 0$

Energy-Momentum Stress Tensor

$T^{ik} = \begin{pmatrix} W & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & \sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & \sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & \sigma_{zz} \end{pmatrix}$
$W = \frac{E^2+B^2}{8\pi}$ $\vec{S} = c\frac{\vec{E}\times\vec{B}}{4\pi}$ $\vec{F} = \oint_S T\cdot d\vec{A} - \frac{d}{dt}\int_V\vec{S}dV$
$T_{ij} = \sigma_{ij} = \frac{1}{4\pi}[\underbrace{E_iE_j + B_iB_j}_{tension} - \frac{1}{2}\delta_{ij}\underbrace{(E^2+B^2)}_{pressure}]$

Conservations

energy:
$\frac{dW}{dt} = \int_V\vec{E}\cdot\vec{J}dV = -\frac{d}{dt}\int_V\left(\frac{E^2+B^2}{8\pi}\right)dV - \oint_A\vec{S}\cdot d\vec{A}$

mechanincal energy density: $\vec{J}\cdot\vec{E}$
EM energy density in V: $\frac{E^2+B^2}{8\pi}$
rate of flow of energy out of V: $\oint_A \vec{S}\cdot d\vec{A}$
momentum:
$\frac{\partial}{\partial t}\left(\vec{P}_{mech} + \int_V \frac{\vec{E}\times\vec{B}}{4\pi c}dV\right) = \oint_A T\cdot d\vec{A}$
field momentum density in V: $\frac{\vec{E}\times\vec{B}}{4\pi c}$
force across surface A: $\oint_A T\cdot d\vec{A}$
field angular momentum density: $\vec{r}\times\left(\frac{\vec{E}\times\vec{B}}{4\pi c}\right)$

Multipoles

electric dipole:
$\phi \simeq \vec{r}\cdot\vec{p}/r^3 \quad \vec{E} = \frac{3\vec{r}\vec{r}\cdot\vec{p}-r^2\vec{p}}{r^5} \quad \vec{E}_{\vec{p}=p\hat{z}} = \frac{p}{r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)$
$\vec{p} = \int\vec{r}'\rho(\vec{r}')dV' = \sum_i q_i\vec{r}_i'$
$\vec{\tau} = \vec{p}\times\vec{E} \quad \vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad U = -\vec{p}\cdot\vec{E}$
$Q_{\alpha\beta} = \int[3x'_\alpha x'_\beta - r'^2\delta_{\alpha\beta}]\rho(\vec{r}')dV' = \sum_i q_i(3x'_{i\alpha}x'_{i\beta} - r_i'^2\delta_{\alpha\beta})$
magnetic dipole:
$\vec{A} \simeq \frac{\vec{m}\times\vec{r}}{r^3} \quad \vec{B} = \frac{3\vec{r}\vec{r}\cdot\vec{m}-r^2\vec{m}}{r^5} \quad \vec{B}_{\vec{m}=m\hat{z}} = \frac{m}{r^3}\left(2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right)$
$\vec{m} = \frac{1}{2c}\int dV\vec{r}'\times\vec{J}(\vec{r}') = \frac{NIA}{c}\hat{n}$ for current loop
$\vec{\tau} = \vec{m}\times\vec{B} \quad \vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) \quad U = -\vec{m}\cdot\vec{B}$
$\vec{m} = \frac{e}{2c}\vec{r}\times\vec{v}$ for moving charge

Electrostatics

$\vec{R} = \vec{r}-\vec{r}'$

vector to field point: $\vec{r}$ vector to source point: $\vec{r}'$
$\vec{E}(\vec{r}) = \int_V\frac{\rho(\vec{r}')}{R^2}\hat{R}dV' \quad \vec{F} = q\vec{E} \quad \phi_E = \int_S\vec{E}\cdot d\vec{A}$
$\phi(\vec{r}) = -\int_\varnothing^{\vec{r}}\vec{E}\cdot d\vec{l} \quad \phi(\vec{r}) = \int\frac{\rho(\vec{r}')}{R}dV'$
$W = \int\frac{E^2}{8\pi}dV = \frac{1}{2}\int\rho\phi dV = \frac{1}{2}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{r}_\alpha-\vec{r}_\beta|}$
in matter: $W_E = \int\frac{\vec{E}\cdot\vec{D}}{8\pi}dV = \frac{1}{2}\int\rho_f\phi dV$

Conductors

a. $\vec{E}=0$ inside
b. $\rho = 0$ inside
c. any net charge resides on surface
d. surface is equipotential
e. $\vec{E}\perp$ to surface

Magnetostatics

$\vec{B} = \frac{1}{c}\int_V\frac{\vec{J}(\vec{r}')\times\vec{R}}{R^3}dV \quad \vec{F} = \frac{I}{c}\int d\vec{l}\times\vec{B} \quad \phi_B = \int\vec{B}\cdot d\vec{A}$
$\vec{K} = \sigma\vec{v} \quad \vec{J} = \rho\vec{v} \quad I = \int_A\vec{J}\cdot d\vec{A}$
$\oint A\cdot d\vec{l} = \int\vec{B}\cdot d\vec{A} = \phi_B$
$W = \int\frac{\vec{H}\cdot\vec{B}}{8\pi}dV = \frac{1}{2c}\int_V(\vec{J}_f\cdot\vec{A})dV$

Induction/Circuits

$\phi = M_{21}I_1c \quad \phi = LIc \quad \epsilon mf = -\int_a^b\vec{E}\cdot d\vec{l} = -L\frac{dI}{dt} \quad W = \frac{1}{2}LI^2$
capacitors:
$C = \frac{Q}{V} \quad W = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$
series: $\frac{1}{C_{eq}} = \frac{1}{C_1}+\frac{1}{C_2}+\ldots$ parallel: $C_{eq} = C_1+C_2+\ldots$
resistors:
$\vec{J} = \sigma\vec{E} \quad V = IR \quad P = IV = I^2R = \frac{V^2}{R}$
series: $R_{eq} = R_1+R_2+\ldots$ parallel: $\frac{1}{R_{eq}} = \frac{1}{R_1}+\frac{1}{R_2}+\ldots$
Kirchhoff's rules:
1. The sum of the currents entering any junction in a circuit must equal the sum of the currents leaving that junction.
2.The sum of the potential differences across all elements around any closed circuit loop must be zero.

Specific Useful Forms

infinite wire: $\vec{E} = \frac{2\lambda}{r}\hat{r} \quad \vec{B} = \frac{2I}{cr}\hat{\phi}$
solenoid: $\vec{B} = \frac{4\pi}{c}nI\hat{z}$
toroidal solenoid: $\vec{B} = \frac{2NI}{cr}\hat{\phi}$

Images

an image is never in the region where it contributes to the potential, it is essentially a new problem where the conductor or dielectric is removed
sphere with total charge Q or potential $V\neq 0$:
$\phi = \frac{q}{|\vec{r}-\vec{a}|} - \frac{qR/a}{\left|\vec{r}-\vec{a}\frac{R^2}{a^2}\right|} + \frac{Q+qR/a}{|\vec{r}|} + \frac{VR}{|\vec{r}|} \quad r>R$

Separation of Variables

$\nabla^2\phi = 0 \quad \nabla^2\phi = -4\pi\rho$
$E,B\to 0$ at ∞ $\quad E,B$ finite at origin
cartesian: $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \to \phi = X(x)Y(y)Z(z)$
likely to get sin, cos, sinh, cosh

cylindrical: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{d\phi}{dr}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2} = 0$
$x = r\cos\theta \quad y = r\sin\theta \quad z = z$
2D case: $\phi(r,\theta) = a_0+b_0\ln r/r_0+\sum_{l=1}^\infty(a_lr^l+b_l/r^l)\cos l\theta + \sum_{l=1}^\infty(a_lr^l+b_l/r^l)\sin l\theta$
grounded conducting cylinder in uniform E-field:
$\phi\sim A\cos(\theta)+\frac{B}{r}\cos(\theta)$
3D case: $\Theta(\theta)\sim\cos(l\theta),\sin(l\theta) \quad Z(z)\sim\sinh(kz),\cosh(kz)$
$R(r)\sim J_l(kr),N_l(kr)$

spherical:
$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\phi^2} = 0$
$x = r\sin\theta\cos\phi \quad y = r\sin\theta\cos\phi \quad z = r\cos\theta$
2D case: $\phi(r,\theta) = \sum_{l=0}^\infty\left(A_lr^l+B_l/r^{l+1}\right)P_l(\cos\theta)$
$P_l(x) = \frac{1}{2^ll!}\left(\frac{d}{dx}\right)^l(x^2-l)^l$
$\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$
grounded conducting sphere in uniform E-field:
$\phi(r,\theta)\sim Ar\cos\theta+\frac{B}{r^2}\cos\theta$
3D case: $\phi(r,\theta,\phi) = \sum_{lm}R_{lm}(r)Y_{lm}(\theta,\phi)$
$P_l(x) = \frac{1}{2^ll!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$
$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{l+m)!}}P_l^m(\cos\theta)e^{im\phi}$

Electromagnetic Waves

$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0 \Rightarrow \vec{E} = \Re\tilde{\vec{E}}e^{i\vec{k}\cdot\vec{r}-i\omega t}$
$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0 \Rightarrow \vec{B} = \Re\tilde{\vec{B}}e^{i\vec{k}\cdot\vec{r}-i\omega t}$
ω set by source (does not change in medium), only k changes in medium
transverse waves $\nabla\cdot\vec{E} = 0$ (polarization defined by $\vec{E}$):
$\vec{k}\cdot\vec{E}_0 = 0 \quad \vec{B}_0 = \hat{k}\times\vec{E}_0 \quad |\vec{E}_0| = |\vec{B}_0|$
Poynting: $\langle\vec{S}\rangle = \frac{c}{8\pi}|\vec{E}_0|^2\hat{k} = \langle W\rangle c\hat{k}$ Intensity: $I = \langle S\rangle$ Energy Density: $\langle W\rangle = \frac{|\vec{E}_0|^2}{8\pi} = \langle\vec{p}\rangle c$
Momentum Density: $\langle\vec{p}\rangle = \frac{|\vec{E}_0|^2}{8\pi}\frac{\hat{k}}{c}$
Radiation Pressure: $P = \frac{I}{c}$
Reflection/Refraction:
normal incidence: $R_\perp = \left|\frac{\sin\theta'-\theta}{\sin\theta+\theta'}\right|^2$
parallel incidence: $R_\parallel = \left|\frac{\tan\theta'-\theta}{\tan\theta+\theta'}\right|^2$
1.incident, reflected, transmitted waves all lie in plane of incidence (defined by normal of the boundary cross k)
2.$\theta_I = \theta_R$
3.$n_I\sin\theta_I = n_T\sin\theta_T$
total internal reflection: $\sin\theta_I = \frac{n_T}{n_I}$
Brewster angle (angle where reflected wave estinguished): $\tan\theta_B\simeq\frac{n_2}{n_1}$
dispersion: $k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)\mu(\omega) \quad n(\omega) = \frac{c}{v_{ph}} = \sqrt{\epsilon\mu}$

Quasistatic Field Near Conductor

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} = \frac{4\pi}{c}\sigma\vec{E}\Rightarrow\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\nabla^2\vec{H} = \frac{4\pi}{c}\sigma\vec{\nabla}\times\vec{E}$
$-\nabla^2\frac{\vec{B}}{\mu} = \frac{4\pi\sigma}{c}\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)\Rightarrow\nabla^2\vec{B} = \frac{4\pi\sigma\mu}{c^2}\frac{\partial\vec{B}}{\partial t}$
skin depth:
1. solve zeroth order problem assuming conductor is perfect
2. use vacuum solution as B.C. for skin depth problem
fields go like $e^{-i\omega t}$
$\delta = \sqrt{\frac{c^2}{2\pi\sigma\omega\mu}} \quad P = \int_0^\infty\sigma\langle E^2\rangle dx$

Waveguides

$\vec{E}, \vec{B} \sim e^{-i\omega t}$ $\vec{\nabla} \times \vec{E} = \frac{i\omega}{c}\vec{B}$ $\vec{\nabla} \times B = -\frac{i\omega}{c}\vec{E}$ $\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \vec{B} = 0$

$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\left[\vec{E}, \vec{B}\right] = 0$

$\vec{E} = \vec{E}(x, y)e^{ikz - i\omega t}$ $\quad \vec{B} = \vec{B}(x, y)e^{ikz - i\omega t}$

$\left[\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right][E_z(x, y), B_z(x, y)] = 0$ $\quad \nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

BC's: $\vec{E}_\parallel = 0 \quad B_\perp = 0$

case 1: TM mode $B_z = 0$ B.C. $E_z = 0$ on boundary

case 2: TE mode $E_z = 0$ B.C. $\frac{\partial B_z}{\partial n} = 0$ on boundary

case 3: TEM mode $B_z = 0$ $E_z = 0$ (must have two conductors)

cut-off frequency: $0 < k^2\omega^2 = \omega^2 - \gamma_{nm}^2 c^2 \to \omega > \omega_c = \gamma_{min} c$

dispersion relation: $\omega(k)$

rectangular geometry:

TM: $E_z(x, y) = E_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

TE: $B_z(x, y) = B_0 \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$

lowest frequency: first non-zero TE mode with k=0

circular geometry:

TM: $E_z = \sum_{m,n} J_m\left(\frac{\gamma_{mn}}{a} r\right)(A_m \cos m\theta + B_m \sin m\theta)$

$\gamma_{mn} = n^{th}$ zero m^{th} order Bessel function

Resonant Cavities

Radiation

Must have $\vec{p}$ (or $\vec{m}$) $= \vec{p}_0 e^{-i\omega t}$

electric dipole:

$\frac{d\langle P\rangle}{d\Omega} = \frac{ck^4}{8\pi}|\vec{p}|^2 \sin^2\theta \Rightarrow \langle P\rangle = \frac{ck^4}{3}|\vec{p}|^2 = \frac{\omega^4}{3c^3}|\vec{p}|^2$

magnetic dipole:

$\frac{d\langle P\rangle}{d\Omega} = \frac{ck^4}{8\pi}|\vec{m}|^2 \sin^2\theta \Rightarrow \langle P\rangle = \frac{ck^4}{3}|\vec{m}|^2 = \frac{\omega^4}{3c^3}|\vec{m}|^2$

General: $P = \frac{2}{3}\frac{|\ddot{\vec{p}}(t)|^2}{c^3} + \frac{2}{3}\frac{|\ddot{\vec{m}}(t)|^2}{c^3} + \frac{1}{180c^5}\sum_{\alpha\beta}|Q_{\alpha\beta}(t)tripledot|^2$

Larmor forumla: $P = \frac{2}{3}\frac{e^2}{c^3}|\dot{\vec{v}}|^2 = \frac{2}{3}\frac{e^2}{m^2c^2}\left(\frac{d\vec{p}}{dt}\right)^2$