Classical Mechanics

Basics

$\sum \vec{F} = m\vec{a}$ $\quad \vec{v} = \vec{\omega} \times \vec{r}$ $\quad a_c = v^2/r = \omega^2 r$
$E_f - E_i = (W_{other})_{i \to f}$ $U_{spring} = \frac{1}{2}kx^2$ $U_{grav} = mgy$ (y-axis up) $\vec{R}_{CM} = \frac{1}{M}\int \rho(\vec{r}')\vec{r}dV$ $\vec{p} = m\vec{v}$ $\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = \vec{p}_f - \vec{p}_i$
$\vec{\tau} = \vec{r} \times \vec{F}$ $\quad \vec{L} = \vec{r} \times \vec{p}$ $\quad \sum \tau_z = I\alpha_z$ $\quad T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$
$P_{fluid} = \rho g h$

Gravitation

$\vec{g} = -G\int_V \frac{\rho(\vec{r}')}{R^2}\hat{R}dV$ $\quad \phi = -G\int_v \frac{\rho(\vec{r}')}{|\vec{R}|}dV$
$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \to \int_A \vec{g} \cdot d\vec{A} = -4\pi G \int_V \rho dV$

Dynamical Systems

$N = 1$ systems:
$\frac{du}{dt} = f(u)$

$$\dot{u} = f(u) = \begin{cases} f(u) > 0 & \dot{u} > 0 \to move\ to\ the\ right \\ f(u) < 0 & \dot{u} < 0 \to move\ to\ the\ left \\ f(u) = 0 & \dot{u} = 0 \to fixed\ point \end{cases}$$

$N = 2$ systems:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} V_x(x,y) \\ V_y(x,y) \end{pmatrix}$$

fixed points: $V_x(x^*, y^*) = 0$ $\quad V_y(x^*, y^*) = 0$ $\quad (\dot{x} = 0\ \ \dot{y} = 0)$

$$M = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} \qquad det(\lambda 1 - M) = \lambda^2 - T\lambda + D$$

$\lambda =$ eigenvalue $\quad T =$ trace $\quad D =$determinanat

$$\text{eigenvalues: } \begin{cases} both+: & unstablenode \\ both-: & stablenode \\ one+one-: & saddle \\ complex: & spiral \\ & stableif \Re < 0 \\ & unstableif \Re > 0 \end{cases}$$

Bifurcations:
saddle node: $\frac{du}{dt} = r + u^2$
transcritical: $\frac{du}{dt} = ru - u^2$
supercritical pitchfork: $\frac{du}{dt} = ru - u^3$
subcritical pitchfork: $\frac{du}{dt} = ru + u^3$
imperfect: $\frac{du}{dt} = h + ru - u^3$

Diagrams

Phase Portraits:
1. Plot y vs. x and identify fixed points eq. ẋ=0 ẏ=0 simultaneously
2. Find nullclines eq. curves where either ẋ=0 or ẏ=0
3. Vectors on phase diagrams are (ẋ, ẏ)

Bifurcation Diagrams:
1. plot u vs. r and draw fixed point equations (line-stable, dotted line-unstable)
2. draw flows from test points, u̇ vs. u graph, ex. if u̇¿0, u increases, arrow up

Linear Oscillations

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = 0$
underdamped: $\omega_0^2 > \beta$
overdamped: $\omega_0^2 < \beta$
critically damped: $\omega_0^2 = \beta$
$\ddot{x} + 2\beta\dot{x} + \omega_0^2 = F(t)$
use Fourier transform on both sides to solve

Nonlinear Oscillators

Multiple time scale analysis:
Relaxation Oscillations:

Lagrangian/Hamiltonian

$$\delta S[q(t)] = \delta \int_{t_a}^{t_b} dt L(q, \dot{q}, t) = 0 \to \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}_{p_\sigma} = \underbrace{\frac{\partial L}{\partial q_\sigma}}_{F_\sigma}$$

$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$ $\quad H(q,p,t) = \sum_\sigma p_\sigma \dot{q}_\sigma - L$
$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$ $\quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma}$ $\quad \frac{dH}{dt} = -\frac{\partial L}{\partial t}$
if $T = T_2 + T_1 + T_0 \to H = T_2 - T_0 + U$

Noether's Theorem: To each and every continuous symmetry of a mechanical system is associated a conserved quantity. Let $\tilde{q}(q, \lambda)$ be a one-parameter family of transformations of the q's, parameterized by λ, with $\tilde{q}_\sigma(q, \lambda = 0) = q_\sigma$, i.e. $\lambda = 0$ identity transformation so $0 = \frac{d}{d\lambda}\Big|_{\lambda=0} L(\tilde{q}, \dot{\tilde{q}}, t) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\frac{\partial \tilde{q}_\sigma}{\partial \lambda}\right)_{\lambda=0}$ so the conserved charge $Q = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\frac{\partial \tilde{q}_\sigma}{\partial \lambda}\Big|_{\lambda=0}$
Holonomic constraint: $G_k(q,t) = 0$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^k \lambda_j \frac{\partial G_j}{\partial q_\sigma} = Q_\sigma$ with $G_j = 0$
Q_σ: generalized force of constraint

Central Forces/Orbital Mechanics

$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$
$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$ $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$ $\quad M = m_1 + m_2$ $\quad \mu = \frac{m_1 m_2}{m_1 + m_2}$
angular momentum conservation: $\vec{l} = \vec{r} \times \vec{p} \to l = \mu r^2 \dot{\phi}$
effective potential: $U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$
$U'_{eff}(r) = 0 \to$ circular orbit
$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{l^2 s^2}F(s^{-1})$ $\quad s = 1/r$ $\quad F(r) = -\frac{\partial U(r)}{\partial r}$
Kepler problem: $U(r) = -kr^{-1}$
$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{l^2 s^2}F(s^{-1}) = \frac{\mu k}{l^2} \to r(\phi) = \frac{r_0}{1 - \epsilon\cos(\phi - \phi_0)}$
where $r_0 = \frac{l^2}{\mu k} = a(1 - \epsilon^2)$ with $a = -\frac{k}{2E}$
circle: $\epsilon = 0, E = -\mu k^2/2l^2, radiusa = l^2/\mu k$
ellipse: $0 < \epsilon < 1, E = -\mu k^2/2l^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1 - \epsilon^2}$
parabola: $\epsilon = 1, E = 0$
hyperbola: $\epsilon > 1, E > 0$
apoapsis: furthest approach, local max
periapsis: closest approach, local min
precession: $\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi(\beta^{-1} - 1)$
where $\beta^2 = \left(\frac{\mu\omega r_0}{l}\right)^2$
escape velocity: $v_{esc}(r) = \sqrt{\frac{2G(M+m)}{r}}$

Conservative Forces

$m\ddot{\vec{r}} = -\vec{\nabla}U(\vec{r}) \equiv \vec{F}(\vec{r})$ $\quad \oint d\vec{r} \cdot \vec{F} = 0$
$W_{ab} = \int_{r_a}^{r^b} d\vec{r} \cdot \vec{F}(\vec{r}) = U(\vec{r}_a) - U(\vec{r}_b) = T_b - T_a$

Accelerated Coordinate Systems

$\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \omega \times \vec{r}$

$$\vec{F}_{eff} = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - \underbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{centrifugal} - \underbrace{2m\vec{\omega} \times \vec{v}_r}_{Coriolis}$$

earth: $\frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}'}{m} + \vec{g} - 2\vec{\omega}\times\frac{d\vec{r}}{dt} - \vec{\omega}\times(\vec{\omega}\times\vec{r})$ where $\vec{g} = -GM_e\hat{r}/r^2$ and $\vec{F}'$ is the sum of all other earthly forces

Rigid Body Motion

$\vec{P} = \sum_i m_i\dot{\vec{r}}_i \quad \dot{\vec{P}} = \vec{F}^{(ext)} \quad \vec{L} = \sum_i m_i\vec{r}_i\times\dot{\vec{r}}_i \quad \dot{\vec{L}} = \vec{N}^{(ext)}$

$I_{\alpha\beta} = \sum_i(\vec{r}_i^2\delta_{\alpha\beta} - r_{i,\alpha}r_{i,\beta}) = \int d^3r\rho(\vec{r})(\vec{r}^2\delta_{\alpha\beta} - r_\alpha r_\beta)$

$I_{\alpha\beta}(\vec{b}) = I_{\alpha\beta}(0) + M(\vec{b}^2\delta_{\alpha\beta} - b_\alpha b_\beta)$ where $I_{\alpha\beta}(0)$ is wrt CM

Principal Axes: $\vec{L} = I_{\alpha\beta}\omega_\beta \to \vec{L} = I_\alpha\omega_\alpha$

1. Find the diagonal elements of I' by setting $\det(\lambda\cdot 1 - I) = 0$.
2. For each eigenvalue λ_b, solve the d equations $I_{\mu\nu}\psi^b_\nu = \lambda_b\psi^b_\mu$
3. Choose normalization $\psi^a_\mu\psi^b_\mu = \delta^{\alpha\beta}$

Euler's equations:

$\vec{N}^{(ext)} = \left(\frac{d\vec{L}}{dt}\right)_{inertial} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega}\times\vec{L} = I\dot{\vec{\omega}} + \vec{\omega}\times(I\vec{\omega})$

$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + N_1^{ext}$

$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + N_2^{ext}$

$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + N_3^{ext}$

solid sphere: $I = \frac{2}{5}MR^2$ solid cylinder: $I = \frac{1}{2}MR^2$

hoop: $I = MR^2$ thin rod, axis through one end: $I = \frac{1}{3}ML^2$

thin rod, axis through center: $I = \frac{1}{12}ML^2$

Coupled Oscillations

$\frac{\partial U}{\partial q_i}\Big|_{q=\bar{q}} = 0 \quad q_i = \bar{q}_i + \eta_i \quad L = \frac{1}{2}T_{ij}\dot{\eta}_i\dot{\eta}_j - \frac{1}{2}V_{ij}\eta_i\eta_j$ where

$T_{ij} = \frac{\partial^2 T}{\partial\dot{q}_i\partial\dot{q}_j}\Big|_{q=\bar{q}} \qquad V_{ij} = \frac{\partial^2 U}{\partial q_i\partial q_j}\Big|_{q=\bar{q}}$

1. find eigenfrequencies from $\det(\omega^2T - V) = 0$
2. find eigenvectors from $(\omega_a^2T_{ij} - V_{ij})\psi^a_j = 0$
3. normalize by $\psi^a_iT_{ij}\psi^a_j = \delta_{\alpha\beta}$

Poisson Brackets

$\{A, B\} = \sum_{\sigma=1}^n\left(\frac{\partial A}{\partial q_\sigma}\frac{\partial B}{\partial p_\sigma} - \frac{\partial A}{\partial p_\sigma}\frac{\partial B}{\partial q_\sigma}\right)$

$\{f, g\} = -\{g, f\} \qquad \{f + \lambda g, h\} = \{f, h\} + \lambda\{g, h\}$

$\{fg, h\} = f\{g, h\} + g\{f, h\} \qquad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

if $\{A, H\} = 0$ and $\frac{\partial A}{\partial t} = 0$, then $\frac{dA}{dt} = 0$, i.e. A is a constant of the motion

if $\{A, H\} = 0$ and $\{B, H\} = 0$, then $\{\{A, B\}, H\} = 0$ and if A and B have no explicit time dependence, then $\{A, B\}$ is a constant of the motion

$\{q_\alpha, q_\beta\} = 0 \qquad \{p_\alpha, p_\beta\} = 0 \qquad \{q_\alpha, p_\beta\} = \delta_{\alpha\beta}$

Canonical Transformations

$q_\sigma = q_\sigma(Q_1, \ldots, Q_n; P_1, \ldots, P_n, t)$

$p_\sigma = p_\sigma(Q_1, \ldots, Q_n; P_1, \ldots, P_n, t)$

to be canonical we want $\dot{Q}_\sigma = \frac{\partial\tilde{H}}{\partial P_\sigma} \quad \dot{P}_\sigma = -\frac{\partial\tilde{H}}{\partial Q_\sigma}$; also Poisson brackets preserved, i.e. $\{Q_\alpha, P_\beta\} = \delta_{\alpha\beta}$

$\tilde{H}(Q, P, t) = H(q, p, t) + \frac{\partial F}{\partial t}$ with

$\frac{\partial F}{\partial q_\sigma} = p_\sigma \qquad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \qquad \frac{\partial F}{\partial p_\sigma} = 0 \qquad \frac{\partial F}{\partial P_\sigma} = 0$

$$F(q, Q, t) = \begin{cases} F_1(q, Q, t) & ; \quad p_\sigma = +\frac{\partial F_1}{\partial q_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \quad (typeI) \\ F_2(q, P, t) - P_\sigma Q_\sigma & ; \quad p_\sigma = +\frac{\partial F_2}{\partial q_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_2}{\partial P_\sigma} \quad (typeII) \\ F_3(p, Q, t) + p_\sigma q_\sigma & ; \quad q_\sigma = -\frac{\partial F_3}{\partial p_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_3}{\partial Q_\sigma} \quad (typeIII) \\ F_4(p, P, t) + p_\sigma q_\sigma - P_\sigma Q_\sigma & ; \quad q_\sigma = -\frac{\partial F_4}{\partial p_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_4}{\partial P_\sigma} \quad (typeIV) \end{cases}$$

Hamilton-Jacobi Theory

$\tilde{H}(Q, P, t) = 0 \to \frac{\partial F}{\partial t} = -H$

$\frac{\partial S}{\partial q_\sigma} = p_\sigma \qquad \frac{\partial S}{\partial t} = -H \qquad \frac{dS}{dt} = L$

$\Gamma_\sigma = \frac{\partial S}{\partial\Lambda_\sigma} = \begin{cases} +Q_\sigma & if\Lambda_\sigma = P_\sigma \\ -P_\sigma & if\Lambda_\sigma = Q_\sigma \end{cases}$

Hamilton-Jacobi eq.: $H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0$

$Q_\sigma(t) = const \qquad P_\sigma(t) = const$

Time-Independent H:

$S(q, \Lambda, t) = W(q, \Lambda) - \Lambda_1 t \to H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \Lambda_1$

Action-Angle Variables

$J = \frac{1}{2}\oint pdq \qquad p = p(\phi, H) \qquad \dot{\theta} = \omega = \frac{\partial H}{\partial J} \qquad T = \frac{2\pi}{\omega} = \frac{\partial J}{\partial H}$

Adiabatic Invariance

Mechanical Mirror:

adiabatic invariant: $J = \frac{1}{2\pi}\oint pdq = \frac{1}{2\pi}\left[\int_{-y(x)}^{y(x)} m|v_y|dy + \int_{y(x)}^{-y(x)} m(-|v_y|)dy\right] = \frac{2}{\pi}m|v_y|y(x)$

Magnetic Mirror:

adiabatic invariant: $\frac{mv_\perp^2}{2|\vec{B}|}$ kinetic energy conserved also

Strings

$\mathcal{L} = \frac{1}{2}\mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2$

$\mu(x)$: mass density $\qquad \tau(x)$: tension

$\frac{\partial}{\partial x}\left[\tau(x)\frac{\partial y}{\partial x}\right] - \mu(x)\frac{\partial^2 y}{\partial t^2} = 0$

τ, μ constant $\to \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \qquad c = \sqrt{\frac{\tau}{\mu}}$

D'Alambert's Solution: $y(x, 0) = \phi(x) \qquad \dot{y}(x, 0) = \psi(x)$

$y(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$

Reflection/Transmission:

e^{ikx} = wave to the right (incident, transmitted)

e^{-ikx} wave to the left (refected)

point mass at $x = 0$ on spring with friction on mass in vertical direction:

$y(x, t)$ continuous at boundary

$m\ddot{y}(0, t) + \gamma m\dot{y}(0, t) = \tau y'(0^+, t) - \tau y'(0^-, t) - 2ky(0, t)$

Membranes

Fluid Mechanics

mass current: $\vec{j} = \rho\vec{v} \qquad \vec{f}_{ext} = \rho\vec{g} = -\rho\nabla\chi$

continuity: $\partial_t\rho + \nabla\cdot\vec{j} = 0 \qquad$ stagnation point: $\vec{v} = 0$

Euler's equation: $\underbrace{\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \frac{\vec{f}_{ext}}{\rho}}_{ideal fluids, inviscid, incompressible}$

incompressible: $\nabla\cdot\vec{v} = 0$

steady or "slow" flow: $\frac{\partial\vec{v}}{\partial t} = 0$

B.C. $\vec{v}\cdot\hat{n} = \vec{V}\cdot\hat{n}$ where $\vec{V}$ is the velocity of the body

vorticity: $\vec{\omega} = \nabla\times\vec{v}$

circulation: $\Gamma(c) = \oint_C \vec{v}\cdot d\vec{l}$

Bernouilli's eq: $\frac{1}{2}v^2 + \frac{p}{\rho} + \chi$ = const along streamlines

Potential Flow

$\vec{v} = \nabla\phi \qquad \vec{\omega} = \nabla\times\nabla\phi = 0 \qquad \nabla\cdot\vec{v} = 0$

2D: $v_x = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \qquad v_y = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$

ψ = stream function, constant along streamlines

$W = \phi + i\psi \qquad \frac{\partial W}{\partial z} = v_x - iv_y = \bar{v}$

$\psi = 0$ on boundaries $\qquad$ B.C. $\vec{v}\cdot\hat{n} = \vec{V}\cdot\hat{n}$

Conformal Mapping

$Z = f(z)$ analytic with inverse $z = F(Z)$, then $w(F(Z)) = W(Z)$ is analytic if $w(z)$ is

$\frac{dw}{dz} = \frac{dZ}{dz}\frac{dW}{dZ} \qquad \bar{V} = \bar{v}\frac{dz}{dZ}$ so in order to have same flow at ∞,

$\frac{dz}{dZ} \to 1$ as $Z \to \infty$

half plane to wedge: $Z = f(z) = z^{\alpha/\pi}$ $\quad z = F(Z) = Z^{\pi/\alpha}$

Blassius's Theorem: $\bar{F} = F_x - iF_y = \frac{i}{2}\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz$

C=contour around body

$\tau = \frac{1}{2}\Re\left\{\rho \oint_C \left(\frac{dw}{dz}\right)^2 z dz\right\}$

inversions:

plane $\Im z = 0$ $\quad g(z) = f(z) + \bar{f(\bar{z})}$

circle $|z| = R$ $\quad g(z) = f(z) + f\left(\frac{R^2}{\bar{z}}\right)$

Viscous Flow

$$\frac{\partial \vec{v}}{\partial t} + \underbrace{(\vec{v} \cdot \nabla)\vec{v}}_{inertialterm} = -\frac{1}{\rho}\nabla p + \underbrace{\nu \nabla^2 \vec{v}}_{viscousterm} + \frac{\vec{f}_{ext}}{\rho}$$

kinematic viscosity: $\nu = \eta/\rho$

Reynold's number: $R \sim \left|\frac{intertialterm}{viscousterm}\right| \sim \frac{UL}{\nu}$

U=char. fluid speed $\quad$ L=char. length of body

$R \ll 1$: viscosity dominates $\quad$ $R \gg 1$: inertia dominates